Group Theory
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Geometric invariants and HNN-extensions

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1. Introduction

For every HNN-extension

$$(*) \qquad H = \langle B, t\,;\, t^{-1}B_1t = B_2\rangle$$

over a base group B and with stable letter t one has the ***associated homomorphism*** $\chi : H \twoheadrightarrow \mathbf{Z}$ given by $\chi(t) = 1$ and $\chi(B) = 0$. Every homomorphism χ of a group G onto $\mathbf{Z}$ can, of course, be regarded as the associated homomorphism of some HNN-decomposition of G; but in many circumstances G has, in fact, an HNN-decomposition over a ***finitely generated base group*** with associated homomorphism χ. This is, for instance, the case when G is finitely presented, see [2].

We call the HNN-extension $(*)$ ***ascending*** if the first associated subgroup B_1 coincides with the base group B, so that the kernel N of the associated homomorphism χ is the union of the ascending chain

$$\ldots \subseteq t^{-1}Bt \subseteq B \subseteq tBt^{-1} \subseteq t^2Bt^{-2} \subseteq \ldots$$

Correspondingly $(*)$ is ***descending*** if $B_2 = B$. It is interesting to know which homomorphisms $\chi : G \twoheadrightarrow \mathbf{Z}$ are associated to an ***ascending*** HNN-decomposition over a ***finitely generated*** base group. This question is answered in [1] in terms of the 'geometric invariant' Σ of G. The Bieri-Neumann-Strebel invariant Σ of a finitely generated group G is a certain subset of the 'character sphere' $S(G)$, by which we mean the set of all equivalence classes $[\chi] = \{\lambda\chi \mid 0 < \lambda \in \mathbf{R}\}$ of non-zero homomorphisms $\chi : G \longrightarrow \mathbf{R}_{\text{add}}$ under multiplication by positive real numbers. We should mention that Σ captures not only the information about ascending HNN-decompositions over finitely generated base groups but also characterizes the finitely generated normal subgroups of G with Abelian quotient.