Group Theory
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Geometric invariants and HNN-extensions

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1. Introduction

For every HNN-extension

$$(*) \qquad H = \langle B, t\,;\, t^{-1}B_1 t = B_2 \rangle$$

over a base group B and with stable letter t one has the ***associated homomorphism*** $\chi : H \twoheadrightarrow \mathbf{Z}$ given by $\chi(t) = 1$ and $\chi(B) = 0$. Every homomorphism χ of a group G onto $\mathbf{Z}$ can, of course, be regarded as the associated homomorphism of some HNN-decomposition of G; but in many circumstances G has, in fact, an HNN-decomposition over a ***finitely generated base group*** with associated homomorphism χ. This is, for instance, the case when G is finitely presented, see [2].

We call the HNN-extension $(*)$ ***ascending*** if the first associated subgroup B_1 coincides with the base group B, so that the kernel N of the associated homomorphism χ is the union of the ascending chain

$$\ldots \subseteq t^{-1}Bt \subseteq B \subseteq tBt^{-1} \subseteq t^2Bt^{-2} \subseteq \ldots$$

Correspondingly $(*)$ is ***descending*** if $B_2 = B$. It is interesting to know which homomorphisms $\chi : G \twoheadrightarrow \mathbf{Z}$ are associated to an ***ascending*** HNN-decomposition over a ***finitely generated*** base group. This question is answered in [1] in terms of the 'geometric invariant' Σ of G. The Bieri-Neumann-Strebel invariant Σ of a finitely generated group G is a certain subset of the 'character sphere' $S(G)$, by which we mean the set of all equivalence classes $[\chi] = \{\lambda\chi \mid 0 < \lambda \in \mathbf{R}\}$ of non-zero homomorphisms $\chi : G \longrightarrow \mathbf{R}_{\text{add}}$ under multiplication by positive real numbers. We should mention that Σ captures not only the information about ascending HNN-decompositions over finitely generated base groups but also characterizes the finitely generated normal subgroups of G with Abelian quotient.

In this paper, I go one step further by investigating the question as to which homomorphisms $\chi : G \twoheadrightarrow \mathbf{Z}$ are associated to an ascending HNN-extension over a *finitely presented base group*. The answer is given in terms of a new geometric invariant ${}^*\Sigma^2$. This is part of a more general concept in my Thesis [6] where I define a chain of *higher geometric invariants*

$$S(G) \supseteq {}^*\Sigma^1 \supseteq {}^*\Sigma^2 \supseteq \cdots \supseteq {}^*\Sigma^k \cdots$$

generalizing ${}^*\Sigma^2$ and the Bieri-Neumann-Strebel invariant $\Sigma = -{}^*\Sigma^1$. The higher geometric invariant ${}^*\Sigma^k$ allows to decide as to whether a given normal subgroup N of G with G/N Abelian is of type F_k, i.e. has an Eilenberg-MacLane complex $K(G,1)$ with finite k-skeleton.

The paper is organized as follows. In §2 we extend homomorphisms $\chi : G \longrightarrow \mathbf{R}$ to valuations v_χ on the Cayley complex $C = C(X;R)$ of a presentation $\langle X;R\rangle$ of G. In §3 we define the geometric invariants ${}^*\Sigma^1$ and ${}^*\Sigma^2$. The combinatorial characterization of ${}^*\Sigma^1$ in terms of certain loops in the Cayley graph of G shows that, up to a sign, ${}^*\Sigma^1$ coincides with the Bieri-Neumann-Strebel invariant. Generalizing this description to dimension 2 we get a combinatorial characterization of ${}^*\Sigma^2$ in terms of simple diagrams over $\langle X;R\rangle$. We use these descriptions to study ascending HNN-extensions with finitely generated base group in §4. We give the proof of the result in [1] in our geometric setting. Then we give necessary and sufficient conditions (in terms of ${}^*\Sigma^2$) for finite presentation of the base group B in an ascending HNN-extension $G = \langle B, t\,;\, t^{-1}Bt \leq B\rangle$.

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2. Characters and valuations on the Cayley complex

2.1 Let G be a finitely generated group, and d the $\mathbf{Z}$-rank of the abelianization G^{ab} of G. A *character* of G is a non-zero homomorphism $\chi : G \longrightarrow \mathbf{R}$ into the additive group of real numbers. Two characters are equivalent if they coincide up to multiplication by a positive real number. $Hom(G,\mathbf{R}) = Hom(G^{ab},\mathbf{R})$ is a d-dimensional real vector space which can be identified with $\mathbf{R}^d$. The equivalence class $[\chi]$ of a character thus is the ray from 0 through χ in $Hom(G,\mathbf{R}) \cong \mathbf{R}^d$. The *character sphere* $S(G)$ of G is defined to be $S(G) = \{[\chi] \mid \chi \in Hom(G,\mathbf{R}) \setminus \{0\}\}$. $S(G)$ is homeomorphic to the unit sphere S^{d-1}.

A character χ with infinite cyclic image is called a *discrete* character. The subset of the rational points of $S(G) \cong S^{d-1}$ consists of the classes of discrete characters and is dense in $S(G)$. For a rational point $[\chi] \in S(G)$ we always find a representative χ with $\chi : G \twoheadrightarrow \mathbf{Z} \subseteq \mathbf{R}$.

A character χ allows us to interpret the ordering of $\mathbf{R}$ in the preimage of χ: attached to each $[\chi] \in S(G)$, we consider the submonoid $G_\chi = \{g \mid \chi(g) \geq 0\}$ of G. G_χ does not depend upon the choice of the representative $\chi \in [\chi]$.

2.2 Let $\langle X; R\rangle$ be a presentation of G where R is a set of cyclically reduced words in the free group $F(X)$ with basis X. We do not assume that X embeds in G, but will not distinguish notationally between *words* in $X^{\pm 1}$, i.e. elements of $F(X)$, and their images in G.

The *Cayley graph* $\Gamma = \Gamma(X)$ and the *Cayley complex* $C = C(X;R)$ of G are defined as follows (see [5], III.4):

The set V of vertices of C is the set G of elements of the group. The set E of edges of C is $G \times X^{\pm 1}$. An edge (g,x), by definition, links the vertex g to gx. [Note that gx here is regarded as an element of G.] The inverse oriented edge is (gx, x^{-1}). We have a labelling function $\varphi : E \longrightarrow X^{\pm 1}$ defined by $\varphi((g,x)) = x$. φ extends multiplicatively to edge paths in C: if $p = e_1 e_2 \ldots e_n$ is an edge path then $\varphi(p) = \varphi(e_1)\varphi(e_2)\cdots\varphi(e_n)$ is a word in $F(X)$. $\varphi(p)$ is reduced if and only if p is a reduced path. p is a loop if and only if $\varphi(p)$ is in the normal closure of R in $F(X)$. The set F of faces of $C(X;R)$ is $G \times R^{\pm 1}$. A face (g,r) has as boundary the loop p_r at g with label $\varphi(p_r) = r$. The inverse of (g,r) is (g, r^{-1}). The 1-skeleton of the Cayley complex C is called the Cayley graph $\Gamma(X)$ of G with respect to the generators X.

2.3 Let χ be a character of G and C the Cayley complex of G in the presentation $G = \langle X; R\rangle$. We extend χ to a valuation v_χ on C:

If $g \in V$ is a vertex of C, we put $v_\chi(g) = \chi(g)$. For an edge $e = (g, x)$ we define $v_\chi(e) = min\{v_\chi(g), v_\chi(gx)\}$. If $p = e_1 e_2 \cdots e_n$ is an edge path beginning at g then the χ-track of p is the sequence

$$(v_\chi(g), v_\chi(g\varphi(e_1)), v_\chi(g\varphi(e_1)\varphi(e_2)), \ldots, v_\chi(g\varphi(e_1)\varphi(e_2)\cdots\varphi(e_n)))$$

and $v_\chi(p)$ is defined to be the minimum of the χ-track of p. Accordingly we denote by $v_\chi(w)$ the minimum of the χ-track of a word w in $X^{\pm 1}$. If (g, r) is a face of C then $v_\chi((g, r))$ is the minimum of the χ-track of the boundary loop of (g, r).

The automorphisms of the Cayley complex C, i.e. automorphisms of the combinatorial 2-complex C which preserve labels, are exactly those induced by left multiplication of G ([5], III.4.1.). G is the group of deck transformations of C.

A valuation v_χ on C extending a character χ has the following property:

$$(*) \quad v_\chi(gc) = \chi(g) + v_\chi(c) \quad \text{for } c \in V \text{ or } c \in E \text{ or } c \in F \text{ and all } g \in G.$$

Remark. The notion of a valuation on the combinatorial Cayley complex $C(X; R)$ is the special case of a more general notation of valuations v_χ extending a character χ of G. Recall that a G-complex is a CW-complex C together with an operation of G by homeomorphisms which permute the cells. If furthermore the stabilizer of each cell is trivial then C is a *free G-complex.*

Let C be a free G-complex and $\chi \in Hom(G; \mathbf{R}) \setminus \{0\}$. A continuous function $v_\chi : C \longrightarrow \mathbf{R}$ is called a valuation on C associated with χ if

(1) $v_\chi(gc) = \chi(g) + v_\chi(c)$ for all $c \in C, g \in G$

(2) $v_\chi(C^0) \subseteq \chi(G)$ [C^0 is the 0-skeleton of C.]

(3) Let $\sigma \subseteq C$ be a cell with boundary $\partial\sigma$ then

$$min\, v_\chi(\partial\sigma) \leq v_\chi(c) \leq max\, v_\chi(\partial\sigma)$$

for all $c \in \sigma$.

If C is the geometric realization of the Cayley complex $C(X; R)$ of a group G then C is the universal cover of the 2-dimensional CW-complex which is usually called the geometric realization of the presentation $\langle X; R\rangle$ of G (see e.g. [4], p.44). A combinatorial valuation on $C(X;R)$ yields by piecewise linear extension a valuation on C.

2.4 A *full* subcomplex C' of the Cayley complex $C = C(X;R)$ of a group G is a subcomplex with the following property: If e is an edge of C or f a face of C and all vertices g of e or of f are in C' then e or f is in C'. A full subcomplex C' of C is determined by the set of vertices of C'.

Let v_χ be a valuation on C associated with the character χ. The *valuation subcomplex* C_v of C is defined to be the full subcomplex of C spanned by the submonoid G_χ, i.e. by $\{g \mid v_\chi(g) \geq 0\} \subseteq V$. We put $C_{v,\lambda} (\lambda \in \mathbf{R})$ for the full subcomplex of C generated by $\{g \mid v_\chi(g) \geq -\lambda\}$ and C_{-v} for the full subcomplex of C spanned by the subset $\{g \mid v_\chi(g) \leq 0\}$ of the vertices V of C. If $\Gamma = \Gamma(X)$ is the Cayley graph of G with respect to the generating set X then Γ_v is the subgraph spanned by G_χ. Γ_v contains those edges (g, x) of Γ for which $v_\chi(g) \geq 0$ and $v_\chi(gx) \geq 0$. Note that $C_{v_\chi} = C_{v_{\chi'}}$ and $\Gamma_{v_\chi} = \Gamma_{v_{\chi'}}$, if χ and χ' are equivalent characters.

3. The geometric invariants $^*\Sigma^1$ and $^*\Sigma^2$

3.1 We keep the notation and conventions of section 2. Recall that the edge path group of a combinatorial 2-complex is isomorphic with the fundamental group of its geometric realization.

Definition. Let G be a finitely generated group, X a finite generating set of G, and $[\chi] \in S(G)$. We put

$[\chi] \in {}^*\Sigma^1 :\Leftrightarrow$ the valuation subgraph Γ_{v_χ} of the Cayley graph $\Gamma(X)$ of G is connected.

Lemma 1. *Let G be a finitely generated group and $[\chi] \in {}^*\Sigma^1$. Then the valuation subgraph $\Gamma_{v_\chi}(Y)$ of $\Gamma(Y)$ is connected for any finite set Y of generators of G.*

Proof. Let X be a finite set of generators of G such that $\Gamma_v(X)$ is connected, and let Y be another finite set of generators. Each $x_i \in X^{\pm 1}$ is expressible as a word w_i in the generators $Y^{\pm 1}$. We fix such expressions and put $\lambda = min\,\{v_\chi(w_i)\}$. Since $\Gamma_v(X)$ is connected, for each vertex h of $\Gamma_v(Y)$ there is an edge path p in $\Gamma(Y)$ connecting 1 and h such that $v_\chi(p) \geq \lambda$. Furthermore, given two vertices h_1, h_2 of $\Gamma_v(Y)$ with $v_\chi(h_1) \geq \mu$ for $i = 1, 2$ and some $\mu \geq 0$, we can find an edge path p' in $\Gamma_v(Y)$ which connects h_1 and h_2 and fulfills $v_\chi(p') \geq \mu + \lambda$. This follows from the fact that G acts on $\Gamma(Y)$ by left multiplication together with property $(*)$ of 2.3. Let g be a vertex of $\Gamma_v(Y)$. Choose $t \in Y^{\pm 1}$ with $\chi(t) > 0$, and $k \in \mathbf{N}$ such that $\chi(t^k) \geq |\lambda|$. Then there is an edge path p_2 connecting t^k and gt^k such that $v_\chi(p_2) \geq 0$. Let p_1 be the edge path on $\Gamma_v(Y)$ corresponding to the word t^k and starting at 1, and p_3 the path with $\varphi(p_3) = t^{-k}$ starting at gt^k. Then $p_1 p_2 p_3$ is an edge path in $\Gamma_v(Y)$ which connects 1 and g, thus $\Gamma_v(Y)$ is connected.

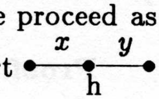
3.2 We give a combinatorial criterion for $[\chi] \in {}^*\Sigma^1$.

Theorem 1 (Criterion for ${}^*\Sigma^1$). *Let G be a finitely generated group, X a finite set of generators, and $[\chi] \in S(G)$.*

Then $[\chi] \in {}^*\Sigma^1$ if, and only if, there is a $t \in X^{\pm 1}$ with $\chi(t) > 0$ such that for every $x \in X^{\pm 1} \setminus \{t, t^{-1}\}$ the conjugate ${}^{t^{-1}}x \in G$ can be expressed as a word w in $X^{\pm 1}$ with $v_\chi(t^{-1}xt) < v_\chi(w)$.

Proof. Let $\Gamma = \Gamma(X)$ be the Cayley graph of G and Γ_v the valuation subgraph of Γ.

If $[\chi] \in {}^*\Sigma^1$ then Γ_v is connected. Take a $t \in X^{\pm 1}$ with $\chi(t) > 0$. Consider the path $t^{-1}xt$ in Γ beginning at 1. If $\chi(x) > 0$ then the endpoint of $t^{-1}xt$ is Γ_v, thus there is also a path w in Γ_v beginning at 1 and ending at $t^{-1}xt$, and therefore $v_\chi(t^{-1}xt) < v_\chi(w)$. If $\chi(x) < 0$ then there exists an integer $l > 0$ such that the endpoint of the path $t^{-1}xt^l$ beginning at 1 lies in Γ_v. Let w' be a word in $X^{\pm 1}$ with $t^{-1}xt^l = w'$ (in G) and $v_\chi(w') > 0$. Thus $t^{-1}xt = w't^{-(l-1)}$ is a desired expression.

Now we consider a vertex g in Γ_v together with a path p connecting 1 and g in Γ. If there is a vertex h in p with $v_\chi(h) < 0$ then we proceed as follows: Choose h such that $v_\chi(h) = v_\chi(p)$, and consider the part 

of p. If $x \neq t, t^{-1}$ and $y \neq t, t^{-1}$ we have expressions ${}^{t^{-1}}x = w_x$ and ${}^{t^{-1}}y = w_y$ with $v_X(t^{-1}xt) < v_X(w_x)$ and $v_X(t^{-1}yt) < v_X(w_y)$. Thus we can pass to a path p' by using the paths labelled by w_x and w_y instead of x and y. If $x = t^{-1}$ or $y = t$ we proceed in the same manner for y or x, respectively, and reduce the path p' afterwards. In any case the number of vertices c with $v_X(c) = v_X(p)$ decreases. Since X is a finite set, $\{v_X(w_x) - v_X(t^{-1}xt) \mid x \in X^{\pm 1} \setminus \{t, t^{-1}\}\}$ has a minimum > 0, and so iteration of the procedure yields eventually a path in Γ_v connecting 1 and g.

Comparing Theorem 1 with [1], Proposition 2.1, it is easy to see that ${}^*\Sigma^1 = -\Sigma$. [$-$ is the antipodal map of $S(G)$.] Hence ${}^*\Sigma^1$ is an open subset of $S(G)$. This follows easily from Theorem 1 too. Note that Bieri, Neumann, Strebel consider G as acting by the right on G', whereas we use left action according to the occidental custom to read edge paths in Γ from left to right.

3.3 A *diagram M* over the presentation $\langle X; R \rangle$ of G is a finite planar configuration of vertices, edges and faces fulfilling the following conditions: The oriented edges of M are labelled by the set $X^{\pm 1}$. If the edge e has label x then its inverse is labelled by x^{-1}. The boundary path of each face of M corresponds under the labelling to a cyclic permutation of a defining relation $r \in R$ or its inverse r^{-1}.

A connected and simply connected diagram M with boundary ∂M a reduced loop p based at 1 describes the equivalence in the edge path group of $C(X; R)$ of p to the trivial path (see [5], III.4 and V.1). We call a connected and simply connected diagram with reduced boundary loop a *simple diagram*.

If a diagram M has two faces which are neighboured as shown in the following illustration

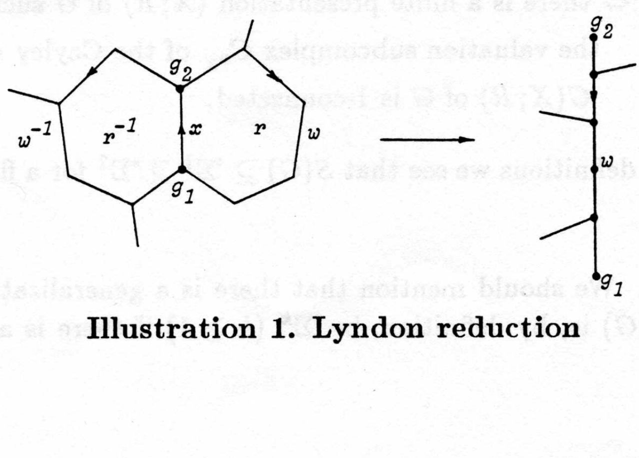


Illustration 1. Lyndon reduction

then we can reduce M by shrinking the interior of the loop labelled ww^{-1}. We refer to this kind of reduction of diagram as *Lyndon reduction.*

Unlike the usual definition of diagrams we do allow trivial faces labelled by $tt^{-1}t^{-1}t$ for a distinguished generator t of G. This deviation simplifies the drawing of diagrams that contain paths coming from conjugation of a word by t.

Each closed path in a simple diagram M is labelled by a relator of G, i.e. a consequence of the defining relations. Thus if we choose a base point of M then every vertex of M can uniquely be labelled by an element of G. For a given valuation v_χ of the Cayley complex $C(X;R)$ of G we get after the choice of a base point in M a *valuated* simple diagram. We denote $v_\chi(M) = min\{v_\chi(g) | g \text{ is vertex in } M\}$.

Obviously, we obtain:

Theorem 2. *Let $G = \langle X; R \rangle$ be a finitely generated group, and C the Cayley complex together with a valuation v_χ associated with the character χ. Then the valuation subcomplex C_v is 1-connected if, and only if, C_v is connected and for each reduced loop p at $1 \in C$ with $v_\chi(p) \geq 0$ there is a simple diagram M with $\partial M = p$ such that $v_\chi(g) \geq 0$ for every vertex g of M.*

3.4. Now we pass to the geometric invariant ${}^*\Sigma^2$.

Definition. Let G be a finitely generated group and $[\chi] \in S(G)$. Then we define

$[\chi] \in {}^*\Sigma^2 :\Leftrightarrow$ there is a finite presentation $\langle X; R \rangle$ of G such that the valuation subcomplex C_{v_χ} of the Cayley complex $C(X;R)$ of G is 1-connected.

From the definitions we see that $S(G) \supseteq {}^*\Sigma^1 \supseteq {}^*\Sigma^2$ for a finitely generated group.

Remark. We should mention that there is a generalization: A character $[\chi] \in S(G)$ is, by definition, in ${}^*\Sigma^k$ $(k \geq 1)$ if there is an Eilenberg-

Maclane complex $K = K(G,1)$ with finite k-skeleton such that the valuation subcomplex C_v of the universal cover C of K is $(k-1)$-connected. [v is a valuation extending χ on the free G-complex C.] [6]

If we change the finite presentation of G, then the valuation subcomplex of the Cayley complex of G will not, in general, remain 1-connected. But a weaker property of the valuation subcomplex is independent of the choice of the finite presentation of G.

Definition. Let C be the Cayley complex of G with respect to the finite presentation $G = \langle X; R\rangle$ and let v_χ be a valuation on C. Suppose C_v is connected. We say that the valuation subcomplex C_v is essentially 1-connected if there is a real number $\lambda \geq 0$ such that the homomorphism $\pi_1(C_v) \longrightarrow \pi_1(C_{v,\lambda})$ induced by the inclusion $C_v \longrightarrow C_{v,\lambda}$ is trivial.

Analogously to Theorem 2, C_v is essentially 1-connected if, and only if, C_v is connected and there exists a $\lambda \geq 0$ such that for every reduced loop based at 1 in C_v we can find a simple diagram M with $\partial M = p$ and $v(g) \geq -\lambda$ for every vertex g of M.

Lemma 2. *Suppose G is a finitely presented group and $[\chi] \in {}^*\Sigma^2$. If $\langle Y; S\rangle$ is a finite presentation of G, then $C_v(Y,S)$ is essentially 1-connected.* [v stands for a valuation on $C(Y,S)$ associated with $[\chi]$.]

Proof. Let $\langle X; R\rangle$ be a finite presentation of G such that $C_v(X;R)$ is 1-connected. We can pass from $\langle X; R\rangle$ to the presentation $\langle Y; S\rangle$ of G by a finite sequence of Tietze transformations. Hence it is sufficient to study the effect of Tietze transformations to the corresponding valuation subcomplexes and to prove that essential 1-connectivity is preserved. Let's fix the following notation:

$$T_1 : \langle X_1; R_1\rangle \longrightarrow \langle X_2; R_2\rangle$$

where $X_2 = X_1$ and $R_2 = R_1 \cup \{r\}$ for a consequence r of R_1.

$$T_2 : \langle X_1; R_1\rangle \longrightarrow \langle X_2; R_2\rangle$$

where $X_2 = X_1 \cup \{y\}$ and $R_2 = R_1 \cup \{r\}$ for a letter $y \notin X_1$ and a relation $r = y^{-1}w$ expressing y as a word w in $X_1^{\pm 1}$.

T_1^{-1} and T_2^{-1} are the transformations in the opposite direction. In both cases, we obviously can view $C(X_1, R_1)$ as a subcomplex of $C(X_2; R_2)$.

(1) If one performs T_1 on $\langle X_1; R_1\rangle$ then $C_v(X_2; R_2)$ is essentially 1-connected, provided that this was the case for $C_v(X_1, R_1)$.

(2) Now we consider T_1^{-1}. Suppose $C_v(X_2, R_2)$ is essentially 1-connected, i.e. for every reduced loop p at 1 in $C_v(X_1, R_1)$ there is a simple diagram M over $\langle X_2; R_2\rangle$ with $\partial M = p$ and $v(g) \geq -\lambda$ for some $\lambda \geq 0$ and for all vertices g of M. Since r is a consequence of R_1 there is a simple diagram M_r over $\langle X_1; R_1\rangle$ with $\partial M_r = r$. Let $\mu = max\{|v_\chi(g) - v_\chi(h)| \mid g, h$ vertices of $M_r\}$. We replace each face of M corresponding to r by M_r and obtain a simple diagram M' with $\partial M' = p$ and $v(g) \geq -\lambda - \mu$ for every g in M'. M' is now a diagram in $C(X_1, R_1)$, hence $C_v(X_1, R_1)$ is essentially 1-connected.

(3) Suppose $C_v(X_1, R_1)$ is essentially 1-connected. Now we perform T_2 on $\langle X_1; R_1\rangle$. Put $\mu = max\{|v_\chi(g) - v_\chi(h)| \mid g, h$ vertices of $r\}$. Each reduced loop based at 1 in $C_v(X_2, R_2)$ is in $C_{v,\mu}(X_2, R_2)$ homotopic to a reduced loop in $C_{v,\mu}(X_1, R_1)$. But $C_{v,\mu}(X_1, R_1)$ is essentially 1-connected, because $C_v(X_1, R_1)$ is so. Therefore $C_v(X_2, R_2)$ is essentially 1-connected.

(4) Suppose $\langle X_1; R_1\rangle$ results from $\langle X_2; R_2\rangle$ by T_2^{-1}, and $C_v(X_2, R_2)$ is essentially 1-connected. For each reduced loop p in $C_v(X_1, R_1)$ we can find a simple diagram M_p over $\langle X_2; R_2\rangle$ with $\partial M_p = p$ and $v(g) \geq -\lambda$ for a $\lambda \geq 0$ and all vertices g in M_p. p has no edge labelled by y or y^{-1}. Since r is the only relation in R_2 involving y, we can remove all occurrences of y (or y^{-1}) in the interior by Lyndon reductions. Thus without loss we can assume that M_p is a diagram over $\langle X_1; R_1\rangle$ and therefore $C_v(X_1, R_1)$ is essentially 1-connected.

Let G be a finitely presented group and suppose that $[\chi] \in {}^*\Sigma^1$. By Theorem 1 there is a finite presentation $\langle X; R\rangle$ of G such that $R \supseteq \{t^{-1}xt = w_x \mid x \in X^{\pm 1} \setminus \{t, t^{-1}\}\}$ where t is a distinguished generator with $\chi(t) > 0$ and $v_\chi(t^{-1}xt) < v_\chi(w_x)$ for all $x \in X^{\pm 1}$, $x \neq t, t^{-1}$. In this situation we can show that $[\chi] \in {}^*\Sigma^2$ implies $\pi_1(C_{v\chi}(X, R)) = 1$.

Lemma 3. *Let G be a finitely presented group and $[\chi] \in {}^*\Sigma^1$. Suppose the presentation $\langle X; R\rangle$ of G contains the defining relations $t^{-1}xt = w_x$ for all $x \in X^{\pm 1} \setminus \{t, t^{-1}\}$ according to Theorem 1. Then we have:*

If $[\chi] \in {}^*\Sigma^2$ then $C_{v_\chi}(X, R)$ is 1-connected.

Proof. Since $[\chi] \in {}^*\Sigma^2$ $C_{v_\chi}(X,R)$ is essentially 1-connected, i.e. for each loop p based at 1 in $C_v(X,R)$ there is a simple diagram M_p with boundary p such that $v_\chi(g) \geq -\lambda$ for all g in M_p and a certain fixed real number $\lambda \geq 0$. Since $R \supseteq \{t^{-1}xt = w_x |$ for all $x \in X^{\pm 1} \setminus \{t, t^{-1}\}\}$ p is freely homotopic in $C_v(X,R)$ to a loop p' based at t^k for $k \in \mathbf{N}$ such that $\chi(t^k) \geq \lambda$. G acts on $C(X,R)$ and the valuation v_χ fulfills $(*)$ of 2.3, thus there is a simple diagram $M_{p'}$ with $\partial M_{p'} = p'$ such that for every vertex g of $M_{p'}$ we have $v_\chi(g) \geq 0$. This implies that $C_v(X,R)$ is 1-connected.

3.5 We assume that G has a presentation $\langle X; R\rangle$ as in Lemma 3. If r is a relation in R, say $r = x_1 x_2 \cdot \cdot x_n$ $(x_i \in X^{\pm 1})$, we write $\hat{r}$ for the word $w_{x_1} w_{x_2} \dots w_{x_n}$, and say that $\hat{r}$ results from r by conjugation by t. [We put $w_x = t$ or t^{-1} if $x = t$ or t^{-1}.] A connected and simply connected diagram with boundary label $\hat{r}$ is denoted by $M_{\hat{r}}$. We want to choose the base point b_1 of $M_{\hat{r}}$ in dependence of the base point b_0 of r, and accordingly we demand for a valuated diagram with boundary r and $M_{\hat{r}}$ in the interior that $v_\chi(b_1) = v_\chi(b_0) + v_\chi(t)$ See Illustration 2.

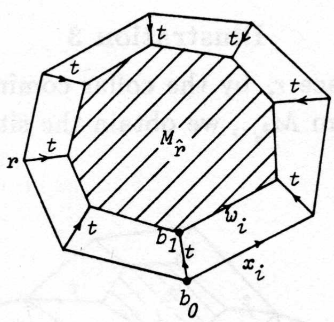


Illustration 2

Theorem 3 (Criterion for ${}^*\Sigma^2$). *Let G be a finitely presented group, and $[\chi] \in {}^*\Sigma^1$. We choose a presentation of G as in Lemma 3. Then $[\chi] \in {}^*\Sigma^2$ if, and only if, for each relation $r \in R^{\pm 1}$ there is a simple diagram $M_{\hat{r}}$ with $\partial M_{\hat{r}} = \hat{r}$ and $v_\chi(r) < v_\chi(M_{\hat{r}})$.*

Proof. If $[\chi] \in {}^*\Sigma^2$ then the valuation subcomplex C_{v_χ} of the Cayley complex C of G with respect to the chosen presentation is, by Lemma 3, 1-connected. Hence for each $\hat{r}$ $(r \in R)$ there is a diagram M based at 1 with $\partial M = \hat{r}$ and $v_\chi(M) \geq v_\chi(\hat{r})$. But we can change the base point and consider M as a diagram M' based at t. Using the notation of Illustration 2, we see that $v_\chi(r) < v_\chi(M')$.

Let p be a reduced loop in the valuation subcomplex C_v based at 1. Since the Cayley complex is 1-connected there is a simple diagram M with $\partial M = p$. If $v_\chi(M) < 0$ we proceed as follows: Let g be a vertex in M with $v_\chi(g) = v_\chi(M)$. For all faces $r_1, r_2, \ldots, r_n$ containing g we have $v_\chi(r_i) = v_\chi(g)$.

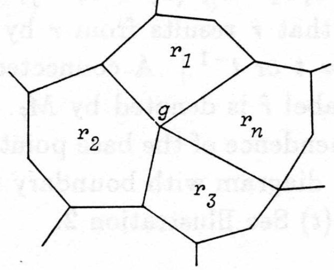


Illustration 3

By replacing each face r_i by the collar coming from conjugation by t together with the diagram $M_{\hat{r}_i}$., we obtain the situation as shown in Illustration 4.

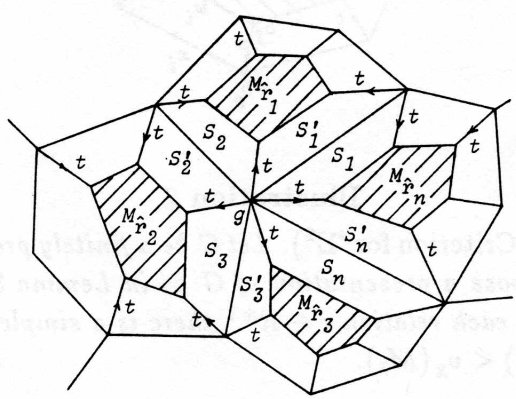


Illustration 4

Obviously s_i and $s_i{}'(1 \leq i \leq n)$ in Illustration 4 are the same relations, but inverse oriented. We can reduce all the faces s_i by Lyndon reductions and the critical vertex g disappears.

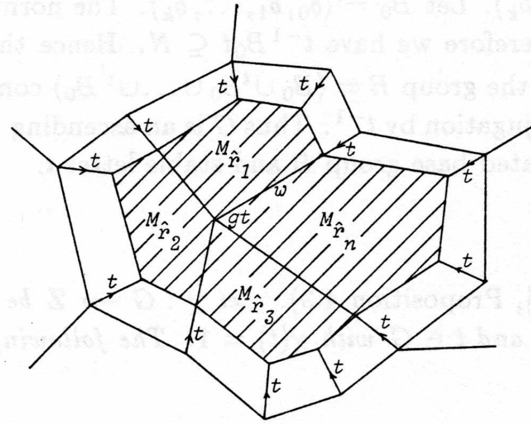


Illustration 5

We get a diagram M_1 which compared with M has only new vertices h with $v_\chi(h) > v_\chi(M)$. By iteration we finally reach a diagram M' with $\partial M' = p$ and $v_\chi(g) \geq 0$ for all vertices $g \in M'$.

As an immediate consequence of Theorem 3 we obtain

Corollary. *Let G be a finitely presented group. Then $^*\Sigma^2$ is an open subset of $S(G)$.*

4. Ascending HNN-extensions with finitely presented base group

Suppose $G = \langle B, t\,;\, t^{-1}Bt = B_2 \leq B \rangle$ is an ascending HNN-extension with finitely generated base group B. For each generator b_i of B, $t^{-1}b_i t$ can be written as a word in the generators of B. The associated character χ_t with $\chi_t(t) = 1$ and $B \subset \operatorname{Ker} \chi_t$ fulfills the condition of Theorem 1, thus $[\chi_t] \in {}^*\Sigma^1$.

If, on the other hand, a rational point $[\chi]$ is in $^*\Sigma^1$ then for a suitable representative χ there is a finite set X of generators of G such that $\chi(t) = 1$ for a distinguished generator t and $\chi(x) = 0$ for the others. We consider the Cayley graph $\Gamma = \Gamma(X)$ of G and denote by Γ_0 the subgraph spanned by the vertices $g \in V$ with $v_\chi(g) = 0$. Γ_0 has as vertices just the elements of $N = \text{Ker } \chi$ and Γ_0 is a subgraph of the connected valuation subgraph Γ_v of Γ. Hence N is finitely generated over the monoid $\langle t \mid$ generated by t, say $N =^{\langle t|} \langle b_0, b_1, \ldots, b_k\rangle$. Let $B_0 = \langle b_0, b_1, \ldots, b_k\rangle$. The normal closure of B_0 in G is N, and therefore we have $t^{-1}B_0t \subseteq N$. Hence there is a positive integer l such that the group $B = \langle B_0 \cup {}^tB_0 \cup \ldots \cup {}^{t^l}B_0\rangle$ contains $t^{-1}B_0t$. B is closed under conjugation by t^{-1}. Thus G is an ascending HNN-extension with finitely generated base group B and stable letter t.

This proves

Theorem ([1], Proposition 4.3). *Let $\chi : G \twoheadrightarrow \mathbf{Z}$ be a discrete character, $N = \text{Ker } \chi$ and $t \in G$ with $\chi(t) = 1$. The following statements are equivalent:*

(i) $[\chi] \in {}^*\Sigma^1$.

(ii) *N is finitely generated as a $\langle t|$-operator group.*

(iii) *G is an ascending HNN-extension $G = \langle B, t\,;\, t^{-1}Bt = B_2\rangle$ with finitely generated base group $B \subseteq N$.*

(iv) *If G is a descending HNN-extension $G = \langle C, t\,;\, t^{-1}C_1t = C\rangle$ and $C \subseteq N$ then $C = N$.*

Let G be a finitely presented group, and $\chi : G \twoheadrightarrow \mathbf{Z}$ an epimorphism. We characterize those $[\chi] \in {}^*\Sigma^1$ for which the base group B in the ascending HNN-extension of the theorem of Bieri, Neumann, Strebel is finitely presented:

We fix the following notation: If $G = \langle b_1, \ldots, b_n, t\,;\, t^{-1}b_it = u_i \ (1 \leq i \leq n), r_1, \ldots, r_m\rangle$ where the u_i and r_j are words in $\{b_1, \ldots, b_n\}^{\pm 1}$ is an ascending HNN-extension, we write $R = \{r_1, \ldots, r_m\}$, $S = \{t^{-1}b_it = u_i \mid 1 \leq i \leq n\}$ and $X = \{b_1, \ldots, b_n, t\}$.

Theorem 4. *Let G be a finitely presented ascending HNN-extension. $G = \langle B, t\,;\, t^{-1}Bt = B_2\rangle = \langle X\,;\, R \cup S\rangle$ with finitely generated base group $B = \langle b_1, \ldots, b_n\rangle$, and χ the associated homomorphism. Then B is finitely presented if, and only if,*

(1) $[\chi] \in {}^*\Sigma^2$ *and*

(2) *there is a finite set $R' \supseteq R$ of words in $\{b_1, \ldots, b_n\}^{\pm 1}$ such that $G = \langle X\,;\, R' \cup S\rangle$ and the component D_{-v} of 1 in $C_{-v}(X, R' \cup S)$ is 1-connected.*

Proof. If B is finitely presented, then there is a finite set $R' \supseteq R$ of words in $\{b_1, \ldots, b_n\}^{\pm 1}$ such that $G = \langle X\,;\, R' \cup S\rangle$ and $\langle b_1, \ldots, b_n\,;\, R'\rangle$ is a finite presentation of B. We use the Cayley complex $C(X\,;\, R' \cup S)$ with respect to this presentation of G. It is easy to see that the associated homomorphism χ fulfills the criterion for $[\chi] \in {}^*\Sigma^2$: Let v be a valuation on C associated with $[\chi]$.

1. If $r \in R'$ then $\hat{r}$ is a word in the generators of B, i.e. a relator of B. Hence there is a diagram $M_{\hat{r}}$ with $v(M_{\hat{r}}) > v(r)$.

2. A defining relation $s \in S$ is of the form $t^{-1}b_i t = u_i$ with $u_i = b_{i_1}^{\varepsilon_1} \ldots b_{i_k}^{\varepsilon_k}$; $(\varepsilon_j = \pm 1)$. We denote the edge path $u_{i_1}^{\varepsilon_1} \ldots u_{i_k}^{\varepsilon_k}$ by $\hat{u}_i$. For all $1 \le i \le n$, we have the diagrams M_i as in Illustration 6.

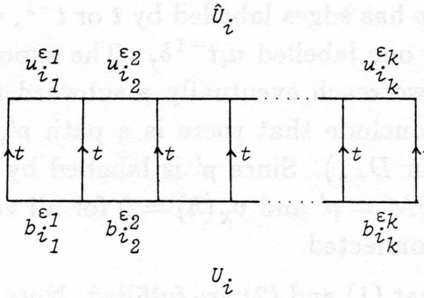


Illustration 6

Now we construct diagrams $M_{\hat{s}}$ for each $s \in S$ such that $v_\chi(s) < v_\chi(M_{\hat{s}})$. See Illustration 7 where $\bar{M}_i$ is the diagram M_i with inverse orientation.

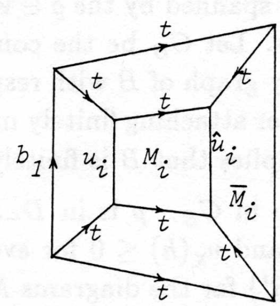


Illustration 7

For the specific choice $u_i = b_2 b_1^{-1} b_3$ e.g., we obtain:

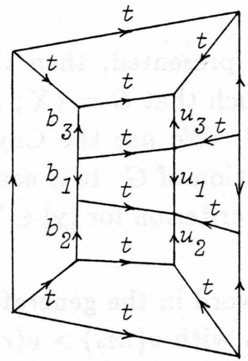


Illustration 8

Let's consider the component D_{-v} of 1 in $C_v(X, R' \cup S)$: We take a loop p at 1 in D_{-v}. If p has edges labelled by t or t^{-1}, we replace a subpath of the form $t^{-1}b_ib_j$ by one labelled $u_it^{-1}b_j$. The exponent sum of t in the label of p is 0, hence we reach eventually a subword $t^{-1}b_kt$ which can be replaced by u_k. We conclude that there is a path p', not containing t or t^{-1}, homotopic to p (in D_{-v}). Since p' is labelled by a relator of B there is a diagram M with $\partial M = p'$ and $v_\chi(h) = 0$ for all vertices h in M. This means that D_{-v} is 1-connected.

We assume now that (1) and (2) are fulfilled. Note that (1) does not assure automatically that the valuation subcomplex C_v of the Cayley complex $C(X, R' \cup S)$ is 1-connected. But since S is a subset of the set of defining relations, Lemma 3 shows that C_v, in fact, is 1-connected.

The subcomplex C_0 of C spanned by the $g \in V$ with $v_\chi(g) = 0$ contains the elements of B as vertices. Let C_B be the component of 1 in C_0. The 1-skeleton of C_B is the Cayley graph of B with respect to the generators b_1, $b_2, \dots, b_n$. We show that - after attaching finitely many relations if necessary - C_B is 1-connected. This implies that B is finitely related.

Let p be a reduced loop in C_B. p is in D_{-v}, thus there is a simple diagram M_1 with $\partial M_1 = p$ and $v_\chi(h) \leq 0$ for every vertex h of M_1. Put $a = \max\{v(h) \mid h \in M_{\hat{r}}, r \in R'\}$ for the diagrams $M_{\hat{r}}$ according to Theorem 3. Since C_v is 1-connected we can proceed as in the proof of Theorem 3 to remove vertices h of M_1 with $v_\chi(h) < 0$. We do so until we reach a diagram

M_2 still in D_{-v} with $\partial M_2 = p$ and $v_\chi(M_2) \geq -a$. M_2 is a diagram in the 'strip' of the Cayley complex limited by $-a$ and 0.

For all $j = 1, 2, \ldots, a$ and all $r \in R'$, let $\hat{r}^{(j)}$ be the word in the generators of B resulting from r by conjugation with t^j. Let $R'' = R' \cup \{\hat{r}^{(j)}\}$. We attach the faces determined by $\{\hat{r}^{(j)}\}$ and get the Cayley complex C'' of G which contains C as a subcomplex. Now it is possible to pass from M_2 to a diagram M' with $v_\chi(h) = 0$ for all vertices h in M'. Hence C''_B is 1-connected.

Corollary. *Let G be a finitely presented group. $\chi : G \twoheadrightarrow \mathbf{Z}$ a discrete character, and $N =$ Ker ; χ. Then N is finitely presented if, and only if, both $[\chi]$ and $[-\chi]$ are in ${}^*\Sigma^2$.*

Proof. Choose $t \in G$ with $\chi(t) = 1$.

Suppose N to be finitely presented, say $N = \langle Y\, ;\, R\rangle$. Put $X = Y \cup \{t\}$ and $S = \{t^{-1}y_i t = u_i \mid y_i \in Y\}$, where the u_i are words in $Y^{\pm 1}$ resulting from y_i by conjugation with t^{-1}. G is the semidirect product $N \rtimes \langle t\rangle$ presented by $G = \langle X\, ;\, R \cup S\rangle$, thus $[\chi] \in {}^*\Sigma^2$. On the other hand G is an ascending HNN-extension with base group N and stable letter t^{-1}, i.e. $[-\chi] \in {}^*\Sigma^2$.

Let $[\chi] \in {}^*\Sigma^2$ and $[-\chi] \in {}^*\Sigma^2$. Since $[\chi]$ and $[-\chi]$ are in ${}^*\Sigma^1$ N is finitely generated, say $N = \langle n_1, n_2, \ldots n_l\rangle$, and therefore G has a finite presentation $\langle n_1, n_2, \ldots n_l, t\, ;\, R\rangle$ such that R includes all relations expressing ${}^{t^{-1}}n_i$ and ${}^{t}n_i$ for $1 \leq i \leq l$ as words in $\{n_1, n_2, \ldots n_l\}^{\pm 1}$. By Lemma 3 $[\chi] \in {}^*\Sigma^2$ implies that the valuation subcomplex C_v of C with respect to this presentation is 1-connected. By the same argument C_{-v}, which coincides with D_{-v}, is 1-connected. Hence N is finitely presented.

Remark. The Corollary above is a special case of the main theorem in [6]: If G is a group of type F_k and N a normal subgroup of G with G/N Abelian then we have

$$N \quad \text{is of type } F_k \Leftrightarrow {}^*\Sigma^k \supseteq S(G, N) = \{[\chi] \in S(G) \mid \chi(N) = 0\}.$$

5. Examples

5.1 Let G be the metabelian group

$$G = \langle a, s, t\,;\ {}^{s^{-1}}a = a^2,\ {}^{t^{-1}}a = a^3,\ [s,t] = 1\rangle$$

and let χ be the epimorphism $\chi : G \twoheadrightarrow \mathbf{Z}$ defined by $\chi(a) = \chi(s) = 0$, $\chi(t) = 1$. The subgroup $B = \langle a, s\rangle$ is the one-relator group $B = \langle a, s\,;\ {}^{s^{-1}}a = a^2\rangle$. By Theorem 4, $[\chi] \in {}^*\Sigma^2$. We can check this easily by writing down the diagrams which are needed for an application of Theorem 3. See Illustration 9.

Furthermore, D_{-v} is 1-connected in this example: Let p be a reduced loop in D_{-v} based at 1 and $\varphi(p)$ the corresponding word. We observe that the exponent sums of t and s in $\varphi(p)$ are zero, and that the χ-track of each initial segment of $\varphi(p)$ is non-positive. Using the relations $t^{-1}at = a^3$ and $t^{-1}st = s$, p is homotopic in D_{-v} to a loop p' such that $\varphi(p')$ is a word in a, a^{-1}, s, s^{-1}. Since the exponent sum of s in $\varphi(p')$ is zero, we can use the relation $s^{-1}as = a^2$ to produce a loop p'' which is homotopic in D_{-v} to p' and the corresponding word $\varphi(p'')$ involves only the letters a and a^{-1}. But the image of $\varphi(p'')$ in G is 1, i.e. the exponent sum of a in $\varphi(p'')$ is zero. Hence p is homotopic in D_{-v} to the trivial loop.

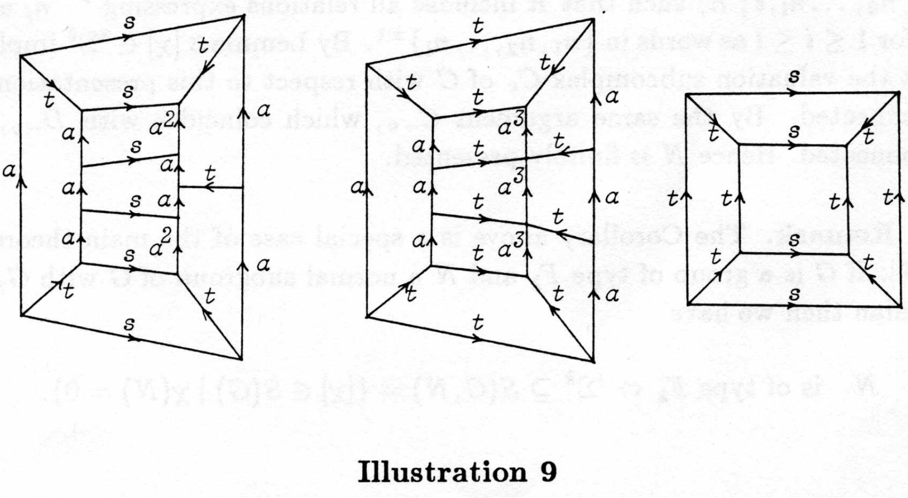


Illustration 9

5.2 Let G be the group in the previous example. Using the criterion for ${}^*\Sigma^2$

and results of Bieri, Strebel [3] about ${}^*\Sigma^1$ of metabelian groups, we calculate in [6] the complement ${}^*\Sigma^{2c}$ of ${}^*\Sigma^2$ for G.

The normal subgroup N of G generated by a and st^{-1} is the kernel of the discrete character χ defined by $\chi(t) = 1$ and $\chi(s) = 1$. We obtain $[\chi] \in {}^*\Sigma^2$ and $[-\chi] \notin {}^*\Sigma^2$. Thus N is not finitely presentable. (See [7] for another argument for this fact.)

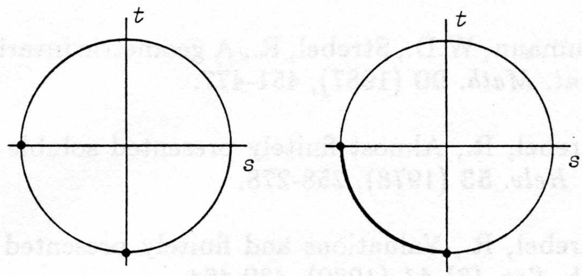


Illustration 10

${}^*\Sigma^{1c}$ and ${}^*\Sigma^{2c}$ for $G = \langle a, s, t\,;\ s^{-1}as = a^2, t^{-1}at = a^3, [s,t] = 1\rangle$.

5.3 In this third example we use Theorem 4 to show that a certain point $[\chi]$ is *not* in ${}^*\Sigma^2$: Let G be the metabelian group of Baumslag and Remeslennikov

$$G = \langle a, s, t\,;\ [a, s^{-1}as] = 1,\ t^{-1}at = as^{-1}as,\ [s,t] = 1\rangle.$$

We consider the character χ with $\chi(t) = 1$ and $\chi(s) = 0$. It is easy to see that $[\chi] \in {}^*\Sigma^1$. Let p be a closed reduced edge path in D_{-v}. We can without loss assume that p has only edges labelled by a, a^{-1} and s, s^{-1}. Since the base group $B = \langle a, s\rangle$ has the presentation $B = \langle a, s\,;\ [a, s^{-j}as^j] = 1, j > 0\rangle$ the label of p is a product of conjugates of these relations in the free group with basis $\{a, s\}$. If for all $i \le n$ a commutes with $s^{-i}as^i$ then

$$1 = t^{-1}[a, s^{-n}as^n]t = [t^{-1}at, s^{-n}t^{-1}ats^n] = [as^{-1}as, s^{-n}as^{-1}ass^n]$$
$$= [a, s^{-(n+1)}as^{(n+1)}]. \qquad \text{(see [7])}$$

Interpreting these equations geometrically we see that for every $j > 0$ there is a simple diagram M with boundary label $\partial M = [a, s^{-j}as^j]$ such that

$v_\chi(g) \leq 0$ for each vertex g of M. Thus p is in D_{-v} homotopic to the trivial loop. Since B being the wreath product of two infinite cyclic groups is not finitely presented, we obtain $[\chi] \notin {}^*\Sigma^2$.

Recently, Bieri and Strebel proved that ${}^*\Sigma^2$ in this example is, in fact, empty.

References

[1] Bieri, R., Neumann, W.D., Strebel, R., A geometric invariant of discrete groups, *Invent. Math.* **90** (1987), 451-477.

[2] Bieri, R., Strebel, R., Almost finitely presented soluble groups. *Comment. Math. Helv.* **53** (1978), 258-278.

[3] Bieri, R., Strebel, R., Valuations and finitely presented groups. *Proc. London Math. Soc.* (3) **41** (1980), 439-464.

[4] Brown, K.S., *Cohomology of groups*, Grad. Texts Math. 87, New York Heidelberg Berlin 1982.

[5] Lyndon, R.C., Schupp, P.E., *Combinatorial Group Theory*, Ergebnisse der Math. und ihrer Grenzgebiete 89, Berlin Heidelberg New York 1977

[6] Renz, B., *Geometrische Invarianten und Endlichkeitseigenschaften von Gruppen*, Thesis, Frankfurt 1988.

[7] Strebel, R., Finitely presented soluble groups, in: *Group Theory - essays for Philip Hall*, edit. K.W. Gruenberg, J.E. Roseblade, London 1984.

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